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SOJOURN IN AN ELLIPTICAL DOMAIN

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We extend results of Maejima (1984) concerning the time that a two-dimensional stationary Gaussian process spends in an elliptical domain. Here: (a) the process may be cross-correlated while the domain is elliptical; (b) the cross-correlations do not vanish asymptotically; (c) a functional limit theorem is obtained.

self-similar process * long-range dependence * multiple Wiener integrals * two-dimensional Gaussian processes * weak convergence

1. Introduction

Maejima (1984) considered a two-dimensional stationary Gaussian process $X(t) = (X_1(t), X_2(t))$ where $X_1(t)$ and $X_2(t)$ are correlated with long-range dependence. He obtained a non-central limit theorem for the time that the process $X(t)$ spends in an elliptical domain D centered at the origin. We extend this result in three ways:

(1) We do not suppose that the ellipse is a circle when $X_1(t)$ and $X_2(t)$ are dependent.

(2) The cross-correlations between $X_1(t)$ and $X_2(t)$ are allowed to have the same order of magnitude as the auto-correlations.

(3) We establish convergence in $C[0,\infty)$, the space of continuous functions on $[0,\infty)$ endowed with the sup-norm topology.

For a recent survey on sojourns, see Maejima (1985).

2. The main result

Let $X(t) = (X_1(t), X_2(t))$, $t \geq 0$, be a two-dimensional stationary Gaussian process with

$$EX(t) = 0, \qquad R(t) \equiv EX(0)^* X(t) = \begin{pmatrix} r(t) & \rho(t) \\ \rho(t) & r(t) \end{pmatrix}$$

where x^* denotes the transpose of the vector x. Suppose in addition that

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$$\begin{aligned}
&r(t) \text{ and } \rho(t) \text{ are continuous,}\\
&r(0)=1, \qquad \rho(0)=\rho \quad (0\le\rho<1),\\
&r(t)\sim t^{-\alpha}L(t) \quad \text{as } t\to\infty,\\
&\rho(t)\sim \rho_\infty t^{-\alpha}L(t) \quad \text{as } t\to\infty,
\end{aligned}$$

where $0<\alpha<\frac{1}{2}$, $0\le\rho_\infty\le 1$ and L is a slowly varying function at infinity.

The components $X_1(t)$ and $X_2(t)$ exhibit a long-range dependence because of the asymptotic form of $r(t)$. These components are said to be asymptotically independent if $\rho_\infty=0$. (Maejima (1984) supposes $\rho_\infty=0$.)

Let

$$D=\{(x_1,x_2)\colon a^2x_1^2+b^2x_2^2\le 1\}, \quad 0<a\le b,$$

be an elliptical domain centered at the origin with principal axes that are parallel to the coordinate axes. Let $I[\cdot]$ be the indicator function and let

$$M(t)=\int_0^t I[X(s)\in D]\,\mathrm{d}s, \quad t>0,$$

denote the amount of time that the process $X(\cdot)$ has spent in the domain D by time t. Our main theorem concerns the asymptotic behavior as $t\to\infty$ of the standardized process

$$Z_t(\tau)=\frac{M(t\tau)-EM(t\tau)}{[\operatorname{Var} M(t)]^{1/2}}, \quad \tau\ge 0.$$

$Z_t(\tau)$ depicts the deviations, adequately rescaled, of the sojourn functional $M(\cdot)$ from its mean $EM(\cdot)$.

Let

$$\begin{aligned}
&A=\tfrac{1}{2}(1+\rho)(a^2+b^2)>0, \qquad B=(1-\rho^2)^{1/2}(a^2-b^2)\le 0,\\
&C=\tfrac{1}{2}(1-\rho)(a^2+b^2)>0,\\
&\alpha_1=\left(\frac{4A}{4AC-B^2}\right)^{1/2}, \qquad \alpha_2=\left(\frac{4C}{4AC-B^2}\right)^{1/2},\\
&\beta_1(y)=\frac{B}{2A}y, \qquad \beta_2(y)=\frac{B}{2C}y,\\
&\gamma_1(y)=\frac{1}{A^{1/2}}\left\{\left(\frac{B^2}{4A}-C\right)y^2+1\right\}^{1/2}, \qquad \gamma_2(y)=\frac{1}{C^{1/2}}\left\{\left(\frac{B^2}{4C}-A\right)y^2+1\right\}^{1/2},\\
&\phi(y)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2},\\
&h_i(y)=(\beta_i(y)-\gamma_i(y))\phi(\beta_i(y)-\gamma_i(y))\\
&\qquad\quad -(\beta_i(y)+\gamma_i(y))\phi(\beta_i(y)+\gamma_i(y)) \quad \text{for } i=1,2,\\
&h_3(y)=\exp\{-\tfrac{1}{2}(\beta_1(y)^2+\gamma_1(y)^2)\}\\
&\qquad\quad \times[\exp\{-\beta_1(y)\gamma_1(y)\}-\exp\{\beta_1(y)\gamma_1(y)\}],
\end{aligned}$$

and finally let

$$c(2,0)=\int_0^{\alpha_1}\phi(y)h_1(y)\,\mathrm{d}y, \tag{2.1}$$

$$c(0,2)=\int_0^{\alpha_2}\phi(y)h_2(y)\,\mathrm{d}y, \tag{2.2}$$

and

$$c(1,1)=\frac{2}{\sqrt{2\pi}}\int_0^{\alpha_1}y\phi(y)h_3(y)\,\mathrm{d}y. \tag{2.3}$$

Let $B_1(\cdot)$ and $B_2(\cdot)$ be two independent Gaussian random measures satisfying $EB_i(A_1)B_i(A_2)=|A_1\cap A_2|$, $i=1,2$, for all Borel sets A_1 and A_2 of $\mathbb{R}$, with $|\cdot|$ denoting Lebesgue measure. Let $\Rightarrow$ denote weak convergence in $C[0,\infty)$.

Theorem. *As* $t\to\infty$,

$$Z_t(\tau)=\frac{M(t\tau)-EM(t\tau)}{(\operatorname{Var}M(t))^{1/2}}\Rightarrow\bar{Z}(\tau)$$

in $C[0,\infty)$. *The limiting process* $\bar{Z}(\tau)$ *admits the following Wiener–Itô double integral representation*:

$$\begin{aligned}\bar{Z}(\tau)=\frac{K(2,\alpha)}{\sigma}\int_{\mathbb{R}^2}f_\tau(y_1,y_2)\Big\{&c(2,0)\frac{1+\rho_\infty}{1+\rho}\,\mathrm{d}B_1(y_1)\,\mathrm{d}B_1(y_2)\\&+c(0,2)\frac{1-\rho_\infty}{1-\rho}\,\mathrm{d}B_2(y_1)\,\mathrm{d}B_2(y_2)\\&+c(1,1)\left(\frac{1-\rho_\infty^2}{1-\rho^2}\right)^{1/2}\mathrm{d}B_1(y_1)\,\mathrm{d}B_2(y_2)\Big\}\end{aligned}$$

with integrand

$$f_\tau(y_1,y_2)=\int_0^\tau\prod_{i=1}^{2}((s-y_i)^+)^{-\alpha/2-1/2}\,\mathrm{d}s, \tag{2.4}$$

and where

$$K(2,\alpha)=\frac{\sqrt{(1-2\alpha)(1-\alpha)}}{\int_0^\infty(x+x^2)^{-\alpha/2-1/2}\,\mathrm{d}x} \tag{2.5}$$

and

$$\sigma^2=2c^2(2,0)\left(\frac{1+\rho_\infty}{1+\rho}\right)^2+2c^2(0,2)\left(\frac{1-\rho_\infty}{1-\rho}\right)^2+c^2(1,1)\frac{1-\rho_\infty^2}{1-\rho^2}. \tag{2.6}$$

Warning. We use the notation $\int_{\mathbb{R}^2}$ in the Wiener–Itô double integral representation for $\bar{Z}(\tau)$, but the definition of that integral can be interpreted as to exclude integration over the diagonal $y_1=y_2$ of $\mathbb{R}^2$.

Remarks. (1) The proof presents an equivalent representation for the limiting process $\bar{Z}(\tau)$ (see (3.4) below). These representations are valid in the sense of the finite-dimensional distributions. See Fox and Taqqu (1984) for further details about moments or cumulants of processes of the type $\bar{Z}(\tau)$.

(2) Maejima (1984) proves the theorem under the assumption $\tau=1$, $\rho_\infty=0$ and $a=b$. When $a=b$ the ellipse is a circle. In that case $c(1,1)=0$ and the cross term vanishes.

(3) Suppose

$$EX_1(s)X_1(s+t)\sim t^{-\alpha_1}L_1(t),$$

$$EX_2(s)X_2(s+t)\sim t^{-\alpha_2}L_2(t),$$

$$EX_1(s)X_2(s+t)\sim EX_2(x)X_1(x+t)\sim t^{-\alpha_3}L_3(t)$$

with α_1, α_2, $\alpha_3>0$ and one of the α_i is smaller than $\frac{1}{2}$. Then the term with the smallest α will provide the main contribution to the variance and hence will dominate in the limit.

(4) The process $\bar{Z}(\tau)$ is *non-Gaussian*, is self-similar with index $H=1-\alpha\in(\frac{1}{2},1)$, that is, the finite-dimensional distributions of $\bar{Z}(a\tau)$ are the same as those of $a^H\bar{Z}(\tau)$ for all $a>0$. The process $\bar{Z}(\tau)$ has also stationary increments, and consequently

$$E\bar{Z}(\tau_1)\bar{Z}(\tau_2)=\tfrac{1}{2}\{\tau_1^{2H}+\tau_2^{2H}-|\tau_1-\tau_2|^{2H}\}. \tag{2.7}$$

(5) If the domain D is an arbitrary compact subset in $\mathbb{R}^2$ in general position, then typically $c(1,0)$ and/or $c(0,1)$ will be non-zero. In that case the main contributions to the limit will be provided by the term of index $n=1$ in the expansion of $M(t)$ in Hermite polynomials (see (3.2) below). Consequently, and for $0<\alpha<1$, the limit of $Z_t(\tau)$ will be the fractional Brownian motion $B_H(\tau)$ with $H=1-\alpha/2\in(\frac{1}{2},1)$. $B_H(\tau)$ is *Gaussian*, has stationary increments, is self-similar with index H and its covariances are also given by (2.7).

3. Proof of the theorem

We shall use the following result whose proof is similar to that of Theorem 6.1 of Fox and Taqqu (1984).

Proposition 1. *Let $(W_1(s), W_2(s))$, $s\geq 0$ be a mean zero, stationary Gaussian vector processes with $EW_1^2(s)=EW_2^2(s)=1$ satisfying*

$$EW_1(s)W_1(s+t)\sim\sigma_1^2t^{-\alpha}L(t),\quad EW_2(s)W_2(s+1)\sim\sigma_2^2t^{-\alpha}L(t),$$

$$EW_1(s)W_2(s+t)\sim\kappa\sigma_1\sigma_2t^{-\alpha}L(t),\quad EW_2(s)W_1(s+t)\sim\kappa\sigma_1\sigma_2t^{-\alpha}L(t),$$

where $\sigma_1^2>0$, $\sigma_2^2>0$, $-1\leq\kappa\leq 1$, $0<\alpha<\frac{1}{2}$, and L is a slowly varying function. Then

$$\frac{\int_0^{t\tau}(W_1(s)W_2(s)-EW_1(s)W_2(s))\,\mathrm{d}s}{t^{1-\alpha}L(t)} \tag{3.1}$$

converges weakly in $C[0,\infty)$ *as* t *tends to infinity to*

$$\frac{1}{\delta}\int_{\mathbb{R}^2} f_\tau(y_1,y_2)\,\mathrm{d}\bar{B}_1(y_1)\,\mathrm{d}\bar{B}_2(y_2) \tag{3.2}$$

where $\bar{B}_1(\cdot)$ *and* $\bar{B}_2(\cdot)$ *are correlated Gaussian random measures satisfying*

$$E\bar{B}_i(A_1)\bar{B}_i(A_2)=\sigma_i^2|A_1\cap A_2|,\quad i=1,2,$$

$$E\bar{B}_1(A_1)\bar{B}_2(A_2)=\kappa\sigma_1\sigma_2|A_1\cap A_2|,$$

for all Borel sets A_1 *and* A_2. *The constant* δ *equals*

$$\delta=\int_0^\infty (x+x^2)^{-\alpha/2-1/2}\,\mathrm{d}s. \tag{3.3}$$

Furthermore, the joint moments of (3.1) *converge to those of* (3.2) *as* $t\to\infty$.

Proof of the theorem. Proceeding first as in Maejima (1984), we orthogonalize the components of $X(t)=(X_1(t),X_2(t))$ by setting $Y(t)=(Y_1(t),Y_2(t))=(X_1(t),X_2(t))T$ where

$$T=\begin{pmatrix}(2(1+\rho))^{-1/2} & (2(1-\rho))^{-1/2}\\ (2(1+\rho))^{-1/2} & -(2(1-\rho))^{-1/2}\end{pmatrix}$$

so that

$$\tilde{R}(t)\equiv EY(0)^*Y(t)=\begin{pmatrix}\dfrac{r(t)+\rho(t)}{1+\rho} & 0\\ 0 & \dfrac{r(t)-\rho(t)}{1-\rho}\end{pmatrix}.$$

But

$$I\{(X_1(s),X_2(s))\in D\}=I\{(Y_1(s),Y_2(s))\in\tilde{D}\}$$

where

$$\tilde{D}=\{(y_1,y_2)\colon (y_1,y_2)T^{-1}\in D\}.$$

Hence,

$$\begin{aligned}M(t)&=\int_0^t I\{(Y_1(s),Y_2(x))\in\tilde{D}\}\\ &=\sum_{n=0}^{\infty}\sum_{n_1+n_2=n} c(n_1,n_2)\int_0^t H_{n_1}(Y_1(x))H_{n_2}(Y_2(x))\,\mathrm{d}s\end{aligned} \tag{3.4}$$

in $L^2(\Omega)$ where

$$\begin{aligned}c(n_1,n_2)&=\frac{1}{n_1!\,n_2!}EI\{(Y_1,Y_2)\in D\}H_{n_1}(Y_1)H_{n_2}(Y_2)\\ &=\frac{1}{n_1!\,n_2!}\int\int_{\tilde{D}} H_{n_1}(y_1)H_{n_2}(y_2)\phi(y_1)\phi(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2,\end{aligned}$$

and where the H_n denote the Hermite polynomials with leading coefficients equal to 1. In particular, $H_0(y)=1$, $H_1(y)=y$, $H_2(y)=y^2-1$.

Maejima (1984) proves that $c(1,0)=c(0,1)=0$, that $c(2,0)$, $c(0,2)$, $c(1,1)$ are given by (2.1), (2.2), (2.3) respectively, that $c(2,0)$ and $c(0,2)$ are non-zero and also that $Z_t(\tau)=[M(t\tau)-EM(t\tau)]/(\operatorname{Var} M(t))^{1/2}$ has same limit as $M'(t\tau)/(\operatorname{Var} M'(t))^{1/2}$ where

$$M'(t)=\int_0^t \{c(2,0)H_2(Y_1(s))+c(0,2)H_2(Y_2(s))+c(1,1)Y_1(s)Y_2(s)\}\,\mathrm{d}s. \tag{3.5}$$

Note that $M'(t)$ has mean 0 and the same asymptotic variance as $M(t)$. To compute the asymptotic standard deviation of $M'(t)$, note that the three terms in the integrand in (3.5) are uncorrelated, and we have

$$EY_1(s)Y_1(s+t)\sim\frac{1+\rho_\infty}{1+\rho}t^{-\alpha}L(t)$$

and

$$EY_2(s)Y_2(s+t)\sim\frac{1-\rho_\infty}{1-\rho}t^{-\alpha}L(t)\quad\text{as } t\to\infty$$

and

$$EH_2(Y_1(s_1))H_2(Y_1(s_2))=2(EY_1(s_1)Y_1(s_2))^2.$$

Therefore

$$(\operatorname{Var} M'(t))^{1/2}\sim\frac{\sigma}{\sqrt{(1-2\alpha)(1-\alpha)}}t^{1-\alpha}L(t)$$

as $t\to\infty$, where σ is defined as in (2.6).

Now set

$$W_1(s)=aY_1(s)+bY_2(s),\qquad W_2(s)=cY_1(s)+dY_2(s) \tag{3.6}$$

where a, b, c, d are constants such that $W_1(s)W_2(s)-EW_1(s)W_2(s)$ has the same distribution as $c(2,0)H_2(Y_1(s))+c(0,2)H_2(Y_2(s))+c(1,1)Y_1(s)Y_2(s)$. The constants a, b, c, d must satisfy the system of equations $ac=c(2,0)$, $bd=c(0,2)$ and $ad+bc=c(1,1)$. Consider for now arbitrary constants $c(2,0)$, $c(0,2)$ and $c(1,1)$ satisfying $4c(2,0)c(0,2)\leqslant c^2(1,1)$. This ensures that the constants a, b, c, d are real-valued. Then

$$EW_1(s)W_1(s+t)\sim\sigma_1^2t^{-\alpha}L(t),$$

$$EW_2(s)W_2(s+t)\sim\sigma_2^2t^{-\alpha}L(t),$$

$$EW_1(s)W_2(s+t)\sim EW_2(s)W_1(s+t)\sim\kappa\sigma_1\sigma_2t^{-\alpha}L(t)$$

where

$$\sigma_1^2 = a^2\frac{1+\rho_\infty}{1+\rho} + b^2\frac{1-\rho_\infty}{1-\rho}, \qquad \sigma_2^2 = c^2\frac{1-\rho_\infty}{1-\rho} + d^2\frac{1-\rho_\infty}{1-\rho},$$

$$\kappa\sigma_1\sigma_2 = ac\frac{1+\rho_\infty}{1+\rho} + bd\frac{1-\rho_\infty}{1-\rho}.$$

Using Proposition 1, we conclude that, as $t\to\infty$,

$$Z_t(\tau) \Rightarrow \bar{Z}(\tau)$$

weakly in $C[0,\infty)$ where

$$\bar{Z}(\tau) = \frac{\sqrt{(1-2\alpha)(1-\alpha)}}{\sigma\delta}\int_{\mathbb{R}^2} f_\tau(y_1, y_2)\,\mathrm{d}\bar{B}_1(y_1)\,\mathrm{d}\bar{B}_2(y_2) \tag{3.7}$$

where δ is defined in (3.3) and σ^2, $\bar{B}_1$, $\bar{B}_2$ are defined in Proposition 1. Note that by (2.5), the multiplicative constant in (3.7) does equal $K(2,\alpha)/\sigma$.

Now let $B_1(\cdot)$ and $B_2(\cdot)$ be two uncorrelated Gaussian measures satisfying $EB_i(A_1)B_i(A_2) = |A_1\cap A_2|$, $i=1,2$. Obtain them by setting

$$\bar{B}_1(A) = a\left(\frac{1+\rho_\infty}{1+\rho}\right)^{1/2} B_1(A) + b\left(\frac{1-\rho_\infty}{1-\rho}\right)^{1/2} B_2(A),$$

$$\bar{B}_2(A) = c\left(\frac{1+\rho_\infty}{1+\rho}\right)^{1/2} B_1(A) + d\left(\frac{1-\rho_\infty}{1-\rho}\right)^{1/2} B_2(A).$$

Then, for arbitrary Borel sets A_1 and A_2,

$$\bar{B}_1(A_1)\bar{B}_2(A_2) - E\bar{B}_1(A_1)\bar{B}_2(A_2)$$

has the same distribution as

$$\begin{aligned} &c(2,0)\left(\frac{1+\rho_\infty}{1+\rho}\right)(B_1(A_1)B_1(A_2) - EB_1(A_1)B_1(A_2)) \\ &\quad + c(0,2)\left(\frac{1-\rho_\infty}{1-\rho}\right)(B_2(A_1)B_2(A_2) - EB_2(A_1)B_2(A_2)) \\ &\quad + c(1,1)\left(\frac{1-\rho_\infty^2}{1-\rho^2}\right)^{1/2} B_1(A_1)B_2(A_2). \end{aligned}$$

Since, for A_1 and A_2 disjoint,

$$E\bar{B}_1(A_1)\bar{B}_2(A_2) = EB_1(A_1)B_1(A_2) = EB_2(A_1)B_2(A_2) = 0,$$

the limiting process $\bar{Z}(\tau)$ admits the equivalent representation

$$\begin{aligned} \bar{Z}(\tau) = \frac{K(2,\alpha)}{\sigma}\int_{\mathbb{R}^2} & f_\tau(y_1,y_2) \\ &\times\left\{c(2,0)\frac{1+\rho_\infty}{1+\rho}\,\mathrm{d}B_1(y_1)\,\mathrm{d}B_1(y_1) + c(0,2)\frac{1-\rho_\infty}{1-\rho}\,\mathrm{d}B_2(y_1)\,\mathrm{d}B_2(y_2)\right. \\ &\quad\left. + c(1,1)\left(\frac{1-\rho_\infty^2}{1-\rho^2}\right)^{1/2}\mathrm{d}B_1(y_1)\,\mathrm{d}B_2(y_2)\right\}. \end{aligned} \tag{3.8}$$

This result has been obtained under the assumption that the constants $c(2,0)$, $c(0,2)$ and $c(1,1)$ belong to the set

$$S=\{(c(2,0), c(0,2), c(1,1)): 4c(2,0)c(0,2)\leq c^2(1,1)\}.$$

It remains to show that the result holds as well when the constants belong to the complement of S in $\mathbb{R}^3$. Since the moments of $M'(t\tau)/(\operatorname{Var} M'(t))^{1/2}$ converge to those of $\bar{Z}(\tau)$ when the constants belong to S, they will also converge when the constants belong to $\mathbb{R}^3\backslash S$ because the convergence of moments is not affected by the values of the constants and because the interior of S is a non-empty open set. In our framework convergence of moments implies convergence of the finite-dimensional distributions (see for example Fox and Taqqu (1984), proof of Proposition 4.1). Since tightness holds as well, this concludes the proof of the theorem. $\square$

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